

In Defense of Picky Eaters: Complexity Separation of Utility and Welfare Maximization Imply Non-Existence of Walrasian Equilibria

Ben Thomas

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1 Introduction

1.1 What makes a "Good" Market, anyway?

We (or, at least I) live in a society, and a major difficulty societies face is how to decide who gets what. What do we do when there are three baguettes, and four people want them? What if there are three people, but one person wants two? What if one of the baguette-wanters is starving? What if the baguette-seller, for whatever reason, is morally opposed to selling baguettes to starving people? Now, what if there are five million baguettes, and three million people want between one and three baguettes, but half are starving, and whether they want one, two, or three baguettes is, for some reason, not related to the fact they are starving? While these examples are silly, they are representative of many of the problems we face designing mechanisms to distribute goods. For starters, there are problems deciding who "ought" to get what. Even if we know there is some relationship between "who"s and "what"s which is best for society out there somewhere, can we actually find out what distribution of goods is? Further, even if we know who ought get what, people have rights over goods which don't necessarily align with who ought get what, so we need to find a way to make people want to exchange their goods so everyone gets what they ought. Frankly, it's a mess. Often so-called optimal distributions do not feel very optimal. For example, in many situations the best economists can offer is "Pareto Optimality", which just means nobody can benefit by trading what they got with what somebody else got. Random serial dictator distribution (everyone picks in a random order) results in a Pareto Optimal result, but anybody with experience with Stanford Housing Assignment

can attest that just because we distributed housing in a Pareto Optimal way doesn't mean that this distribution strikes anybody in it as effective, nor good.

1.2 Why Walrasian Equilibria and Welfare Maximization

Luckily, in many situations there is a very magic feeling result: if we let a market pick the prices of goods and we let people autonomously try to get whatever collection of goods makes them the most happy, we end up with a situation where everyone gets the set of good they are most happy with. This is a very strong sense of optimality (nobody could do better unless there are different prices; if there are different prices society collectively would be less happy). Such a situation is a Walrasian Equilibria.

Admittedly, in many practical situations, this is not especially comforting, since finding a Walrasian Equilibrium is known to be PPAD-complete, and further, even determining whether a Walrasian Equilibria exists at all is an NP-complete problem (McLennan). Further, the kind of information an economist would need to set prices to achieve the Walrasian Equilibria is extraordinarily difficult to get, since anybody interesting in buying an object has numerous reasons to lie to you about that object's worth.

Despite these numerous shortcomings, these results are still useful. Many exchanges are small enough that finding a Walrasian Equilibrium is possible, and while difficult, finding ϵ -Walrasian results has worked well in recent US Spectrum auctions (Milgrom).

As a result, it is very natural to ask under what conditions a Walrasian Equilibrium is known to exist. Frustratingly, with the notable exception of the "Gross Substitutes" condition (Gul and Stacchetti), essentially no general theorems exists, and most results have analyzed markets on a case-by-case basis. Why is this problem so hard? Reproducing the work of Tim Roughgarden, complexity theory will show us not only why this task is difficult, but also give us a better understanding of when such equilibrium exists. Counter intuitively, this proof shows that when people have relatively simple preferences, a Walrasian Equilibria does not exist.

2 Complexity Separations Imply the Non-Existence of Guaranteed Walrasian Equilibria

2.1 Walrasian Equilibria: Formally

We consider a situation where some seller has m items. Note that we are not concerned with the seller's welfare, and the seller will sell any item for any $p > 0$. While this assumption may feel unrealistic, its inclusion will substantially simplify the math without changing the results. Now, let there be k players, with valuations $v_1(S), v_2(S), \dots, v_k(S)$, given a bundle S of the m items. We consider an allocation $\mathbf{A}$, a vector of the bundle purchased by each player. These factors collectively are considered a market, M . We will let each item have a price, p , set exogenously, and represented by the vector $\mathbf{p}$, and we assume all players can spend as much money as they want, although this makes them unhappy, such that their welfare is given by $u_i = v_i(S_i) - p(S_i)$.

Definition 2.1. (Walrasian Equilibrium). A Walrasian Equilibrium for a market M is an allocation $\mathbf{A} = \{S_1, S_2, \dots, S_k\}$ of the m items to k players with valuation functions $v_1, v_2, \dots, v_k$ and a supporting vector of non-negative prices $\mathbf{p} = [p_1, p_2, \dots, p_m]$ such that:

- (W1) All buyers are as happy as possible with their respective allocations, given the prices. That is, for every buyer $i = 1, 2, \dots, k$, we have that $S_i \in \arg \max_S \{v_i(S) - \sum_{j \in S} p_j\}$.
- (W2) The allocation is feasible. That is, no single good is allocated to two buyers. Formally, $S_i \cap S_j = \emptyset$ for $\forall i \neq j$.
- (W3) The market clears. That is, all goods are allocated to some buyer such that for every $j \in M$, $j \in S_i$ for some i .

Now, we prove the important fact that a Walrasian equilibrium maximizes two quantities, subject to prices: individual utility, given by the valuation of each player's bundle sans the bundle's cost, and welfare, which is the collective utility of all players.

Theorem 2.2. (First Welfare Theorem). If prices $\mathbf{p} = \{p_1, p_2, \dots, p_m\}$ and allocation $\mathbf{A} = \{S_1, S_2, \dots, S_m\}$ are a Walrasian equilibrium then

$$\mathbf{A} \in \operatorname{argmax}_{T_1, T_2, \dots, T_k} \sum_{i=1}^k v_i(T_i)$$

where $(T_1, T_2, \dots, T_k)$ ranges over every feasible allocation of goods. The First Welfare theorem yields that under Walrasian Equilibria, there is no "better" distribution of goods, subject to $\mathbf{p}$.

Proof. Let $(S_1^*, S_2^*, \dots, S_k^*)$ denote a welfare maximizing allocation and $(S_1, S_2, \dots, S_k)$ denote the allocation that results from Walrasian Equilibrium. By property W1 we have that, for players $i = 1, 2, \dots, k$:

$$v_i(S_i) - \sum_{j \in S_i} p_j \geq v_i(S_i^*) - \sum_{j \in S_i^*} p_j$$

Now, to check the total welfare, we sum over the k players, yielding that:

$$\sum_{i=1}^k v_i(S_i) - \sum_{i=1}^k \sum_{j \in S_i} p_j \geq \sum_{i=1}^k v_i(S_i^*) - \sum_{i=1}^k \sum_{j \in S_i^*} p_j$$

Now, by W2 and W3 we have that each item is awarded once such that the second terms on both sides resolve to the sum of the prices of the goods, $\sum_{j=1}^m p_j$, which can then be subtracted from both sides. This yields that:

$$\sum_{i=1}^k v_i(S_i) \geq \sum_{i=1}^k v_i(S_i^*)$$

Thus, we have that $\mathbf{A} = (S_1, S_2, \dots, S_k)$ must be welfare maximizing and by extension that all allocations yielded under Walrasian equilibrium are welfare maximizing.

2.2 Statement of Result

This proof is originally the Tim Roughgarden, as cited. I have rephrased parts in order to, in my opinion, aid clarity, and I have elaborated some proofs, however.

Definition 2.3. (Utility Maximization Problem). Given a valuation function, $v_i(S)$, a set of all possible goods m , a bundle $(S) \subseteq M$, and a supporting vector of non-negative prices $\mathbf{p} = [p_1, p_2, \dots, p_m]$, return the bundle which satisfies:

$$v_i(S) \in \max_{S' \subseteq M} [v_i(S') - \sum_{j \in S'} p_j]$$

Definition 2.4. (Utility Maximization Oracle). Given a valuation function, $v_i(S)$, a set of all possible goods m , a bundle $(S) \subseteq M$, and a supporting vector of non-negative prices $\mathbf{p} = [p_1, p_2, \dots, p_m]$, return whether the bundle S satisfies:

$$v_i(S) \in \max_{S \subseteq M} [v_i(S) - \sum_{j \in S} p_j]$$

Definition 2.5. (Welfare Maximization Problem). Given a market with k players, with valuations $v_1, v_2, \dots, v_k$, and utility-maximization oracle, find an allocation $\mathbf{A} = \{S_1, S_2, \dots, S_k\}$ of the m items to the k players, with a supporting vector of non-negative prices $\mathbf{p} = [p_1, p_2, \dots, p_m]$ such that:

$$\sum_{i=1}^k v_i(S_i) = \operatorname{argmax}_{T_1, T_2, \dots, T_k} \sum_{i=1}^k v_i(T_i)$$

Where $T_1, T_2, \dots, T_k$ ranges over every feasible allocation of goods.

Theorem 2.6. (Complexity Separations Imply Non-Existence of Walrasian Equilibria). (Rough-garden) Let $\mathbf{V} = \{v_1, v_2, \dots, v_k\}$ denote a vector of valuation functions for k players. Suppose the welfare-maximization problem for $\mathbf{V}$ can be shown to be strictly harder than the utility-maximization problem for $\mathbf{V}$. Then there exist markets with player evaluations $\mathbf{V}$ which have no Walrasian Equilibrium.

2.3 Proof Sketch

The proof precedes as follows. We assume that $P \neq NP$. In most interesting cases, the Welfare Maximization problem is at least NP-hard. Previous work has shown that, given the existence of a Walrasian Equilibrium, the associated linear programming problem has an optimal integral solution. We show that this solution can be found based on polynomial calls to the Utility Maximization problem. This implies that a Walrasian Equilibrium exists if and only if the Welfare Maximization Problem is no more than polynomially more difficult than the Utility Maximization Problem. Thus, unless we find a proof that $P = NP$, when there is a complexity separation between the Welfare Maximization Problem and the Utility Maximization Problem, the LP solution must not exist, and by extension, there must not be a guaranteed Walrasian Equilibrium.

2.4 Proof of Result

First, consider a general market with k buyers with value functions $v_i(S_i)$, over m goods distributed as bundles, S_i . The following linear program, known as the "configuration LP", finds the welfare maximizing allocation, $\mathbf{A}$.

$$\max \sum_{i=1}^k \sum_{S \subseteq M} v_i(S) x_{iS}$$

such that $\sum_{i=1}^k \sum_{s:j \in S} x_{iS} \leq 1$ for $j = 1, 2, \dots, m$

and $\sum_{S \subseteq M} x_{iS} = 1$ for $i = 1, 2, \dots, k$

where $x_{iS} = \begin{cases} 1 & \text{if } i \text{ gets the bundle } S \\ 0 & \text{otherwise} \end{cases}$

Lemma 2.7. (Configuration LP I). *The welfare-maximizing feasible allocation to any combinatorial allocation problem is yielded by the configuration linear program.*

I will not include a proof of this fact, as the proof is slightly tedious and the reader can quickly convince themselves of this fact by inspection of the linear program. The first term finds the allocation which maximizes welfare, subject to the two "feasibility" constraints. The first enforces that each item is only awarded once, and the second enforces that each buyer only receives one bundle. Note that fractional results are permitted.

Lemma 2.8. (Configuration LP II). *The configuration LP has an exponential number of variables and a polynomial number of constraints.*

Proof. First I show the number of variables is exponential. The maximization part of the configuration LP tests every possible allocation, $\mathbf{A} = \{S_1, S_2, \dots, S_k\}$. However, each S can be any subset of m goods. To count the number of possible bundles we can partition m goods into, we use Euler's Pentagonal Number Theorem:

$$p(m) = \sum_{k=1}^{\infty} (-1)^{k+1} \left(p\left(m - \frac{k(3k-1)}{2}\right) + p\left(m - \frac{k(3k+1)}{2}\right) \right)$$

For simplicity, we will use the following approximation:

$$p(m) \approx \frac{1}{4m\sqrt{3}} e^{\pi\sqrt{\frac{2m}{3}}}$$

Now, as $m \rightarrow \infty$, growth is dominated by the exponential factor, so we have that Euler's Pentagonal Number Theorem is $\approx O(c^{\sqrt{m}})$, which is exponential. This assures us that even in the best case, where all buyers are identical, that this is at least exponential.

Now, consider the constraints. We have to check whether constraint 1 has been violated for m items across i players. Further, we have to check whether constraint 2 has been violated by checking i players. These constraints are polynomial and linear, respectively.

Now, a useful fact about linear programming is that for any linear programming formula (a "primal"), it is possible to construct a "dual", which constrains the results of the primal by minimizing the constraints, subject to new constraints constructed based on the original maximization function. Doing so yields the following dual function, where u_i corresponds to the primal constraint that bidder i receives one bundle and p_j corresponds to the constraint that item j is allocated at most once:

$$\min \sum_{i=1}^k u_i + \sum_{j=1}^m p_j$$

$$\text{such that } u_i + \sum_{j \in S} p_j \geq v_i(S) \text{ for all } i = 1, 2, \dots, k \text{ and } S \subseteq M$$

$$p_j \geq 0 \text{ for } j = 1, 2, \dots, m$$

Corollary 2.9. Reversal of Order of Constraints and Variables in Dual. See lemma 2.8 to see that S increases exponentially with respect to m , but that u_i and p_j increase polynomial with respect to k and j and linearly with respect to j , respectively. By inspection, we can see that the dual has exponential constraints and polynomial variables.

Now, since we have polynomial variables and exponential constraints, we can use the ellipsoid method to solve this. The ellipsoid method is known to be effective at solving linear programming problems in time polynomial to the number of variables, given access to a separation oracle, which we have already assumed to exist. Now, by inspection of the dual, we can see that a separation oracle would require us to solve:

$$u_i \geq \max_{S \subseteq M} [v_i(S) - \sum_{j \in S} p_j]$$

Now, clearly, this is the utility-maximization problem! Thus, we have shown that, subject to there being a solution to the configuration LP for $\mathbf{V}$ that there is a polynomial-time reduction from $\mathbf{V}$'s welfare-maximization problem to its utility-maximization problem.

Theorem 2.10. (Dual Configuration LP). The dual configuration LP can be solved in polynomial calls to a utility-maximization oracle.

Proof. We know that the ellipsoid method can solve a LP problem with polynomial variables and exponential calls to a separation oracle. By corollary 2.9 we have that the dual configuration LP fits this criteria. Further, by definition, we have that the separation oracle and the utility-maximization oracle solve the same problem. Thus, The dual configuration LP can be solved in polynomial calls to a utility-maximization oracle.

Now, finally, we prove that solving the dual LP problem produces a solution to the primal if and only if there is a Walrasian equilibrium. We can solve a primal LP with a dual LP when "complementary slackness" conditions hold. In our case, those conditions are:

1. $x_{iS} > 0$ implies that $u_i = v_i(S) - \sum_{j \in S} p_j$
2. $p_j > 0$ implies that $\sum_i \sum_{S: j \in S} x_{iS} = 1$

However, this is precisely our definition of Walrasian Equilibria from 2.1! This means that our polynomial time solution to the dual program is only a solution to the primal program when the market admits a Walrasian equilibria.

Now, we have that, if and only if there is a Walrasian equilibrium, that the Welfare Maximization problem reduces to the Utility Maximization Problem. Thus, we have our result. If, for a given valuation profile, V , it can be shown that such a reduction does not exist because the Welfare Maximization problem is strictly harder, this then means that there must be market conditions for which V has no Walrasian equilibria.

3 Conclusion

3.1 Final Thoughts

Thus, we have a very counterintuitive result: markets are guaranteed a better optima when the preferences of the individuals in the market are more complicated, and this admits many thought experiments where we can take players with simple utilities, arbitrarily increase them, and then find that there is, as a result, now a guaranteed equilibrium. Unfortunately, this proof has had very limited penetration into economics, and as far as I know has only been referenced in a single paper with Roughgarden wrote with one of his graduate students ("Why Prices need algorithms"). This is unfortunate, because I suspect this proof would be quite useful in economics, and there may even be others like it.

Second, there is a bit more work to be done on extensions of this proof. A, perhaps silly, approach to trying to prove $P = NP$ would be to try lots of markets, and see if the reduction works. If it does, that would be a proof! A formal proof that this is guaranteed to never work strikes me as not especially hard, but probably would be useful. Further, economics includes other concepts of Equilibrium, which may, on inspection, have similar qualities.

Finally, after mulling it over, I decided to not provide concrete examples in this paper. This is because there is not a very clean relationship between what utility functions are actually simple and what functions look simple, and as a result the original exposition I wrote (and subsequently cut), took almost two pages. I decided that it would be best to leave the intuition that simple utility functions don't admit Walrasian equilibria, and not worry about the examples.

3.2 References

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